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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZN 92/79 NOVEMBER

A.M. COHEN

A NEW PARTIAL GEOMETRY WITH PARAMETERS $(s,t,\alpha) = (7,8,4)$

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A new partial geometry with parameters $(s,t,\alpha) = (7,8,4)$

by

Arjeh M. Cohen

ABSTRACT

A partial geometry with parameters as given in the title is constructed by use of the 240 points closest to the origin in the lattice E_8.

KEYWORDS & PHRASES: *partial geometry, strongly regular graph, lattice* E_8.

INTRODUCTION.

A *partial geometry* $(V,\mathcal{L})$ *with parameters* (s,t,α) is a finite nonempty set V of *points* together with a family $\mathcal{L}$ of subsets of V called *lines* such that

(i) For any two points in V there is at most one line containing them both (If such a line exists, the two points are called *collinear*);
(ii) Each line in $\mathcal{L}$ contains exactly s+1 points $(s\geq 1)$;
(iii) Each point is contained in exactly t+1 lines $(t\geq 1)$;
(iv) For any point $x \in V$ and any line $L \in \mathcal{L}$ not containing x there are exactly α points on L collinear with x.

A partial geometry $(V,\mathcal{L})$ with parameters (s,t,α) consists of $v = (s+1)(st+\alpha)/\alpha$ points and $b = (t+1)(st+\alpha)/\alpha$ lines. Furthermore, there are exactly $k = s(t+1)$ points collinear with a given point and $\mu = \alpha(t+1)$ points collinear with each of any two given mutually non-collinear ones.

From $(V,\mathcal{L})$ a graph $G = (V,E)$ can be constructed on the points of V such that two points are adjacent whenever they are collinear. Such a graph is strongly regular with parameters (v,k,λ,μ), where $\lambda = t(\alpha-1)+s-1$ is the number of points collinear with each of any two given collinear ones. This amounts to saying that each point in the graph G has valency k and that any two connected (non-connected) points have $\lambda(\mu)$ common neighbors. More about partial geometries can be found in BOSE [3]. It will be clear that for any system $(V,\mathcal{L})$ not necessarily satisfying all axioms (i),...,(iv) a graph $G = G(V,\mathcal{L})$ can be constructed in the way described above. This graph may very well be strongly regular while $(V,\mathcal{L})$ is not a partial geometry. However, the following holds.

LEMMA. *If* $(V,\mathcal{L})$ *is a pair consisting of a finite nonempty set* V *and a family* $\mathcal{L}$ *of subsets of* V *such that for given* $s,t \geq 1$, *the axioms* (i), (ii), (iii) *are satisfied and such that there is a natural number* α *for which the graph* $G(V,\mathcal{L})$ *is strongly regular with parameters* $(v,k,\lambda,\mu,)$ $= ((s+1)(st+\alpha)/\alpha, s(t+1), t(\alpha-1)+s-1,\mu)$, *then* $(V,\mathcal{L})$ *is a partial geometry with parameters* (s,t,α).

PROOF. It suffices to check axiom (iv). Fix a line $L \in \mathcal{L}$. For $x \in V$ outside L we denote by α_x the number of points in L that are collinear with x. By counting arguments, we obtain

$$\sum_{x \notin L} \alpha_x = (s+1)ts \text{ and}$$
$$\sum_{x \notin L} \binom{\alpha_x}{2} = \binom{s+1}{2}(\lambda-s+1),$$

whence $\sum_{x \in L} (\alpha-\alpha_x)^2 = 0$. This implies that $\alpha_x = \alpha$ for any $x \notin L$, so we are through. □

For a description of the selfdual unimodular lattice E_8 of rank 8, the reader is referred to [2] or [4]. Consider the strongly regular graph G_o whose points are the 120 lines through the origin containing a nonzero vector of minimal distance to the origin in the lattice E_8 (points being connected whenever they represent mutually orthogonal lines with respect to the bilinear form on E_8). Mathon suggested that study of G_o, whose parameters are $(v,k,\lambda,\mu) = (120,63,30,36)$, might lead to a new partial geometry with parameters $(s,t,\alpha) = (7,8,4)$. This note is concerned with the construction of such a partial geometry. The help of H.A. Wilbrink has been crucial for the outcome.

CONSTRUCTION. S_5 (A_5) denotes the symmetric (alternating) group on 5 letters. Moreover a formal element τ outside A_5 is chosen so as to obtain a copy $\tau A_5 = \{\tau x | x \in A_5\}$ of A_5. Now put $V = A_5 \cup \tau A_5$. We shall use the permutation representations c of S_5 on V by conjugation and r of A_5 on V by right multiplication, both acting in such a way that τ is fixed. Thus $c(g)(\tau a) = \tau g a g^{-1}$ $(a \in A_5, g \in S_5)$ and $r(h)(\tau a) = \tau a h$ $(h,a \in A_5)$. Conjugation of $v \in V$ by $g \in S_5$ will also be denoted by writing g in the exponent of v, i.e. $v^g = c(g)v$. Similarly for subsets X of V:

$$X^g = \{x^g | x \in X\}$$

We write down three lines of V explicitly:

$L_1 = \{1,(15243),(13254),(12345),\tau(23)(15),\tau(34)(25),\tau(13)(45),\tau(124)\}$,
$L_5 = \{1,(12)(34),(13)(24),(14)(23),\tau(142),\tau(243),\tau(134),\tau(123)\}$,
$L_6 = \{1,(14)(25),(12)(45),(24)(15),\tau,\tau(14)(25),\tau(12)(45),\tau(24)(15)\}$.

Finally, denoting by K the group generated by (124) and (14)(25) (isomorphic to A_4), we can define the set L of all lines on V:

$$L = \{L_m^g h \mid g \in K;\ h \in A_5;\ m = 1,5,6\}.$$

Clearly by construction $c(K).\ r(A_5)$ is a group of automorphisms of (V,L) isomorphic to $A_4 \times A_5$. This is not all of Aut (V,L) as for instance

$$\pi\begin{cases} x & \mapsto \tau x^{(1245)} \\ \tau x & \mapsto x^{(1245)} \end{cases} \qquad (x \in A_5)$$

defines an automorphism not contained in this subgroup. Let A be the group of automorphisms generated by $c(K).\ r(A_5)$ and π.

<u>THEOREM</u>. (V,L) *is a partial geometry with parameters* $(s,t,\alpha) = (7,8,4)$. *The corresponding graph is isomorphic to* G_0.

<u>PROOF OF THE THEOREM</u>. First of all we shall establish a correspondence between the points closest to the origin in E_8 and the points of V. In order to do so we present E_8 in the following way. Take $\tau = (1+\sqrt{5})/2$ and consider the skew field $\mathbb{H}(\tau)$ of real quaternions with coefficients in $\mathbb{Q}(\tau)$. Choose the basis 1,i,j,k such that $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. For any $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\tau)$ $(x_0,x_1,x_2,x_3 \in \mathbb{Q}(\tau))$, the conjugate $\bar{x}$, the norm N(x) and the real part Re(x) are defined by $\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$,

$$N(x) = x\bar{x} \quad \text{and}$$

$$\mathrm{Re}(x) = \tfrac{1}{2}(x + \bar{x}) = x_0 \quad \text{respectively.}$$

Thus any nonzero $x \in \mathbb{H}(\tau)$ has inverse $\bar{x}\, N(x)^{-1}$. The subgroup of the multiplicative group of nonzero elements in $\mathbb{H}(\tau)$ generated by i,j, $\tau = \frac{1}{2}(-1 + (1-\tau)i - \tau j)$ will be denoted by Ic. It is isomorphic to $Sl_2(5)$ and of order 120. In fact there is an epimorphism $Ic \to A_5$ determined by $i \mapsto (12)(34)$, $j \mapsto (13)(24)$, $\zeta \mapsto (124)$, $w \mapsto (235)$. Thus each point in A_5 can (and will) be identified with the two points in its inverse image under the epimorphism. In order to extend this identification to all of V, we just identify $\tau x (x \in A_5)$ with $\pm \tau x$, a set of two elements in τIc.

Next we will supply the subring $\mathbb{Z}[Ic]$ of $\mathbb{H}(\tau)$ generated by all elements of Ic with the structure of a $\mathbb{Z}$-lattice by defining a quadratic form q on $\mathbb{Z}[Ic]$. Write $t(a+b\tau) = a$ for $a,b \in \mathbb{Q}$. The form q is then given by $q(x) = 2(t \circ N(x))$ for $x \in \mathbb{Z}[Ic]$. The corresponding bilinear form is $(x,y) = 2t(\mathrm{Re}\, \bar{x}y)$ $(x,y \in \mathbb{Z}[Ic])$.

Now $(\mathbb{Z}[Ic],\underline{q})$ is an even unimodular 8-dimensional lattice, and therefore isomorphic to E_8 (cf. [4], p.55). Moreover each element in V determines a unique line through the origin containing two nonzero points in E_8 closest to the origin, and vice versa.

It is easily checked that if two points $x,y \in V$ are collinear in (V,L), they are perpendicular with respect to the bilinear form derived from q. Thus $G(V,L)$ is a subgraph of G_0 with the same number of points and the same number of edges and therefore coincides with G_0. As to the proof of the first statement of the theorem, clearly each line contains 8 points. There are exactly 9 lines containing the point 1, namely 4 in the A-orbit of L_1, 4 in the A-orbit of L_5, and L_6. They are denoted by $L_1, L_2, L_3, L_4, L_5, L_7, L_8, L_9$ and L_6 respectively and written out explicitly in table 1. As no point $\neq 1$ occurs twice in this table, axioms (i), (iii) hold if one of the points concerned is 1. But the group A acts transitively of the 120 points of V, so the two axioms hold without restriction. Finally, as $G(V,L)$ is strongly regular, axiom (iv) is a consequence of the Lemma. □

<u>REMARKS</u> (i) Let Ω be the subset of A_5 consisting of all elements in the A_5-conjugacy classes of (12345) and (12)(34). The complete subgraph of G_0 on the points of A_5 is the Cayley graph $\Gamma(A_5,\Omega)$ in the BIGGS' notation [1]. If Ω_1 is the union of the A_5-conjugacy classes (12354) and

table 1

The lines in L containing 1

line	elements in the line							
L_1	1	(15243)	(13254)	(12345)	τ(23)(15)	τ(34)(25)	τ(13)(45)	τ(124)
L_2	1	(13425)	(12453)	(14352)	τ(12)(34)	τ(24)(35)	τ(13)(25)	τ(145)
L_3	1	(14523)	(15324)	(13542)	τ(13)(24)	τ(14)(35)	τ(23)(45)	τ(152)
L_4	1	(15432)	(12534)	(14235)	τ(14)(23)	τ(34)(15)	τ(12)(35)	τ(254)
L_5	1	(12)(34)	(13)(24)	(14)(23)	τ(142)	τ(243)	τ(134)	τ(123)
L_6	1	(14)(25)	(12)(45)	(24)(15)	τ.1	τ(14)(25)	τ(12)(45)	τ(24)(15)
L_7	1	(15)(23)	(12)(35)	(13)(25)	τ(132)	τ(235)	τ(125)	τ(315)
L_8	1	(23)(45)	(25)(34)	(24)(35)	τ(234)	τ(354)	τ(245)	τ(253)
L_9	1	(14)(35)	(15)(34)	(13)(45)	τ(143)	τ(135)	τ(154)	τ(345)

table 2

The lines in L containing τ

line	elements in the line							
$L_{\tau 1}$	(142)	(13)(25)	(23)(45)	(15)(34)	τ	τ(14325)	τ(13452)	τ(15423)
$L_{\tau 2}$	(15)(23)	(12)(34)	(14)(35)	(245)	τ	τ(14532)	τ(15234)	τ(12435)
$L_{\tau 3}$	(34)(25)	(13)(24)	(12)(35)	(154)	τ	τ(15342)	τ(12543)	τ(13524)
$L_{\tau 4}$	(13)(45)	(14)(23)	(24)(35)	(125)	τ	τ(13245)	τ(14253)	τ(12354)
$L_{\tau 5}$	(124)	(234)	(143)	(132)	τ	τ(12)(34)	τ(13)(24)	τ(14)(23)
$L_{\tau 6}$	1	(14)(25)	(12)(45)	(15)(24)	τ	τ(14)(25)	τ(12)(45)	τ(15)(24)
$L_{\tau 7}$	(135)	(253)	(123)	(152)	τ	τ(13)(25)	τ(15)(23)	τ(12)(35)
$L_{\tau 8}$	(432)	(254)	(235)	(345)	τ	τ(23)(45)	τ(25)(34)	τ(24)(35)
$L_{\tau 9}$	(134)	(145)	(354)	(153)	τ	τ(15)(34)	τ(13)(25)	τ(14)(35)

(12)(34), then $\Gamma(A_5,\Omega_1)$ is isomorphic to the complete subgraph of $G(V,L)$ on the points of τA_5. Finally, for $x,y \in A_5$ the points $x,\tau y$ of V are joined in $G(V,L)$ if and only if $xy^{-1} \in \Omega_2$, where Ω_2 is the union of the A_5-conjugacy classes of 1,(12)(34) and (123).

(ii) In view of (i) it will be clear that verification G_0 is strongly regular and therefore the proof of the first statement of the theorem could be done without using quaternions or E_8.

(iii) A is a subgroup of $\mathrm{Aut}(V,L)$ of order $2^5.3^2.5$. On the other hand, $\mathrm{Aut}(V,L)$ is a subgroup of $\mathrm{Aut}(E_8)/\{\pm I\}$, so the order of $\mathrm{Aut}(V,L)$ must divide $2^{13}.3^5.5^2.7$.

Note that $\mathrm{Aut}(V,L)$ cannot be all of $\mathrm{Aut}(G_0)$, since the latter group acts transitivily on the family of 15 × 135 8-cliques in G_0, while no more than 135 8-cliques of G_0 originate from lines in L.

(iv) Consideration of parameters might lead to the expectation that the choice of an appropriate sub-family L_0 of lines in L provides a partial geometry on V with parameters $(s,t,\alpha) = (7,4,2)$ or (even weaker) a strongly regular graph with parameters $(v,k,\lambda,\mu) = (120,35,10,10)$. However no selection of A-orbits from L leads to such a family L_0.

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